DEPARTMENT OF PHYSICS & ASTRONOMY

SECOND YEAR LAB REPORT

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EXPERIMENT E7: STUDY OF AN OSCILLATING SYSTEM DRIVEN INTO RESONANCE

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Abstract

This report will investigate the frequency response of an LCR circuit, in terms of amplitude and phase behaviour, as a function of the damping resistance in the circuit. The frequency of the amplitude resonance and its variation with the damping resistance will be studied. The quality factor Q as a function of damping will be determined. The phase difference between the driving voltage and the circuit response as a function of frequency of the driving voltage for high and low damping will be analysed.

1 Contents

2 Introduction

2.1 The LCR Circuit

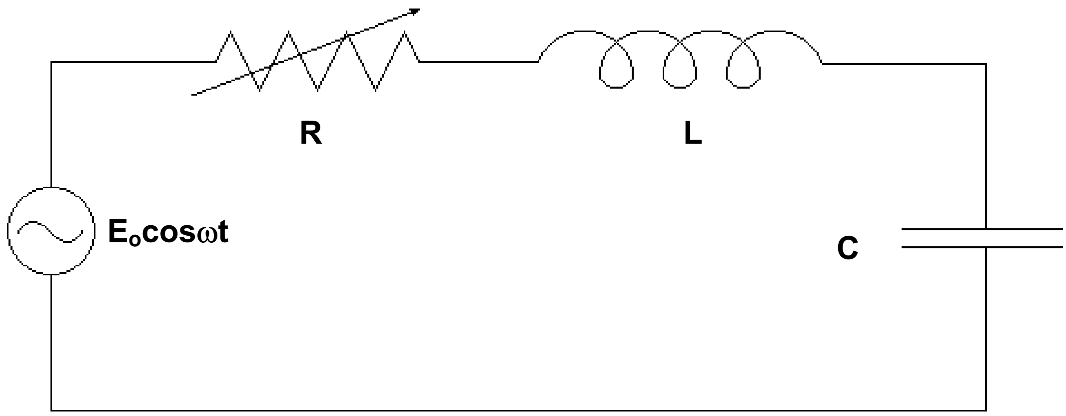


Diagram 1: An LCR circuit

An LCR circuit (a circuit with inductance L, capacitance C and resistance R – see diagram 1) oscillates in simple harmonic motion (SHM) in response to an AC driving voltage. The response of the circuit can be quantified by measuring the voltage across the inductor or the capacitor, the capacitor is chosen in this investigation. Applying Kirchoff's second law to each of the components, it can be shown that any time t, the charge q on the capacitor is given by:

$$L(d^2q/dt^2) + R(dq/dt) + q/C = E_o cos(\omega t) \tag{1}$$

Where E_o is the Amplitude of the driving voltage and ω is the angular frequency of the driving voltage. ω is related to the linear frequency f by:

$$\omega = 2\pi f \tag{2}$$

Equation 1 has the solution of the form:

$$q = q_o cos(\omega t + \phi) \tag{3}$$

Where q_o is the maximum charge that appears on the capacitor and ϕ is the phase angle between the voltage across the capacitor and the driving voltage. This shows that the response of the circuit oscillates with the same frequency as the driving voltage, but at a phase difference of ϕ (see section 2.2 for more detail on the phase relationship).

The amplitude of the voltage across the capacitor is given by:

$$V_c = E_o / C \sqrt{[\omega^2 R^2 + L^2(\omega^2 - \omega_o^2)^2]} \tag{4}$$

Where ω_o is the natural frequency of the circuit, and can be calculated using:

$$\omega_o = \sqrt{(1/LC)} \tag{5}$$

2.2 Phase Relationship Between the Driving and Response Voltages

The signals of both the driving voltage from the oscillator and the response voltage across the capacitor can be plotted on the same axis. Diagram 2 shows this for cases where the frequency of the oscillator is less than the natural frequency of the circuit (diagram 2a), where they are equal (diagram 2b) and where the oscillator frequency is greater than the natural frequency (diagram 2c).

Diagram 2a: $\omega < \omega_o$

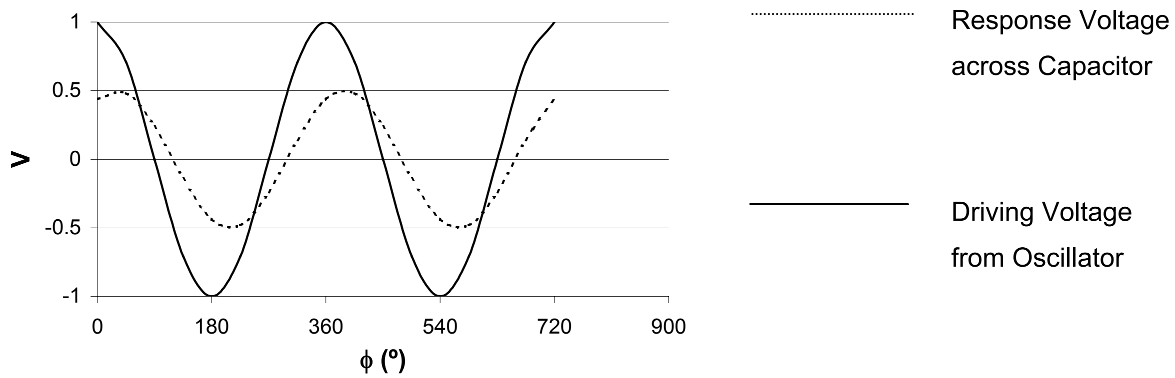


Diagram 2b: $\omega = \omega_o$

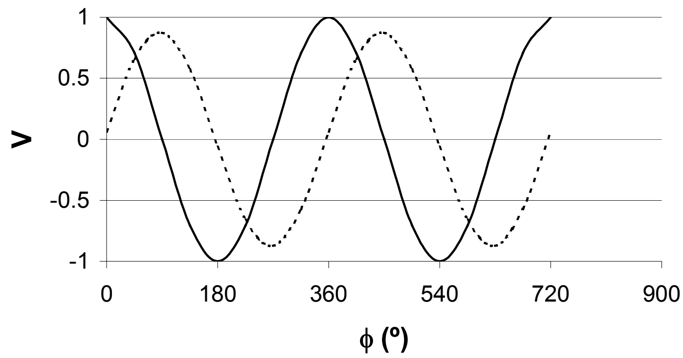
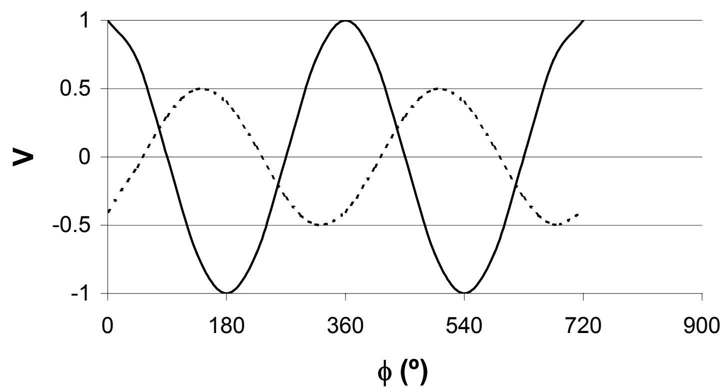


Diagram 2c: $\omega > \omega_o$



From these diagrams it can be seen that for $\omega < \omega_o$, the two signals are nearly in phase, for $\omega = \omega_o$ the two signals are 90° or $\pi/2$ radians out of phase, and for $\omega > \omega_o$ the phase difference is 180° or π radians.

2.3 Resonance

For the case of $\omega \approx \omega_o$ i.e. when the frequency of the driving voltage is near to the natural frequency of the circuit, resonance will occur. Resonance is the term used to describe maximum amplitude of oscillation, i.e. when the response of the circuit is at its maximum value.

By differentiating equation 4, the angular frequency at which resonance occurs is found to be:

$$\omega_{max}^2 = \omega_o^2 - R^2/2L^2 \qquad (6)$$

This shows that in the presence of a damping resistance, resonance does not occur at the natural frequency, but at a lower frequency depending on the damping resistance R.

By substituting $\omega = \omega_o$ into equation 5, the magnitude of the voltage across the capacitor at resonance is found to be:

$$V_c^{(res)} = E_o / \omega_o RC \qquad (7)$$

And hence it can be seen that the magnitude of this maximum also depends upon the damping resistance R.

2.4 The Q Factor

The effect of this is that for lower damping, the amplitude resonance curve (V_c against ω) is sharper. The sharpness of the curve is quantified by the Q factor, which is defined to be:

$$Q = \omega/\Delta\omega = f/\Delta f \qquad (8)$$

Where $\Delta\omega$ is the width of the resonance curve between two points, ω_1 and ω_2 at which:

$$V_c = V_c^{(res)}/\surd 2 \qquad (9)$$

These points correspond to the frequencies at which the power being absorbed by the circuit is half that of the maximum, i.e. they are the half power points. f and Δf are the corresponding linear frequencies, given by equation 2. The Q factor is also defined by:

$$Q = (1/R)\sqrt{(L/C)} \qquad (10)$$

3 Study of amplitude resonance as a function of damping

3.1 Apparatus Set-up

For this investigation an LCR circuit with the following specification is used:

- $L = 30\ mH\ \pm 2\%$
- $C = 1000\ pF\ \pm 2\%$
- *R is variable from $0 - 10K\Omega\ \pm 0.1\%$*
- *Variable frequency oscillator with range $1Hz$ to $100Khz$ and amplitude $12V$*
- *Frequency meter with range $20Hz$ to $10Mhz \pm 0.001\%$*
- *Oscilloscope with accuracy $\pm 3\%$ on both axis*

See diagram 3 for the set-up:

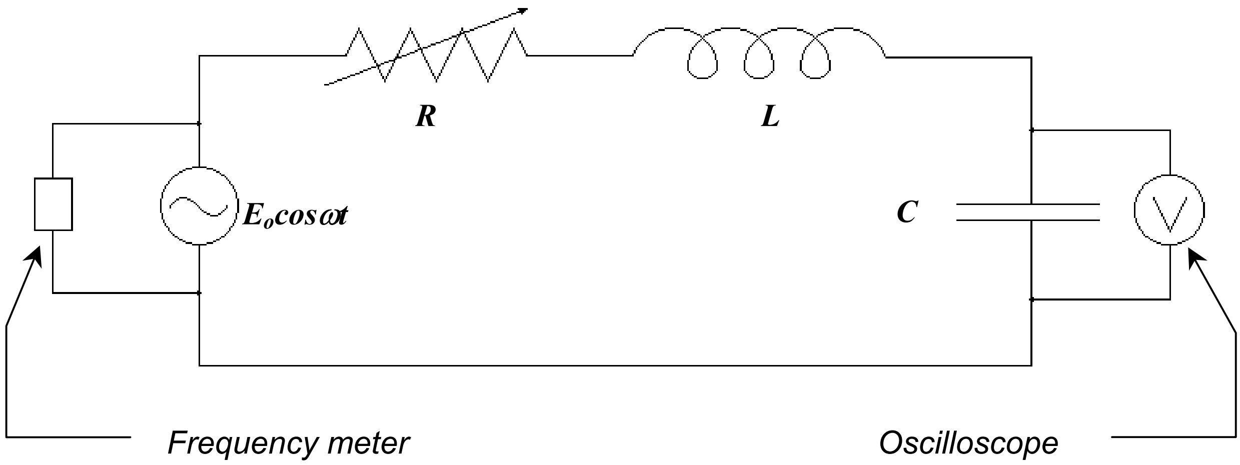


Diagram 3: The circuit set-up

Using the values of L, C and R given above, an estimate for the natural frequency was calculated using equation 5 and equation 2:

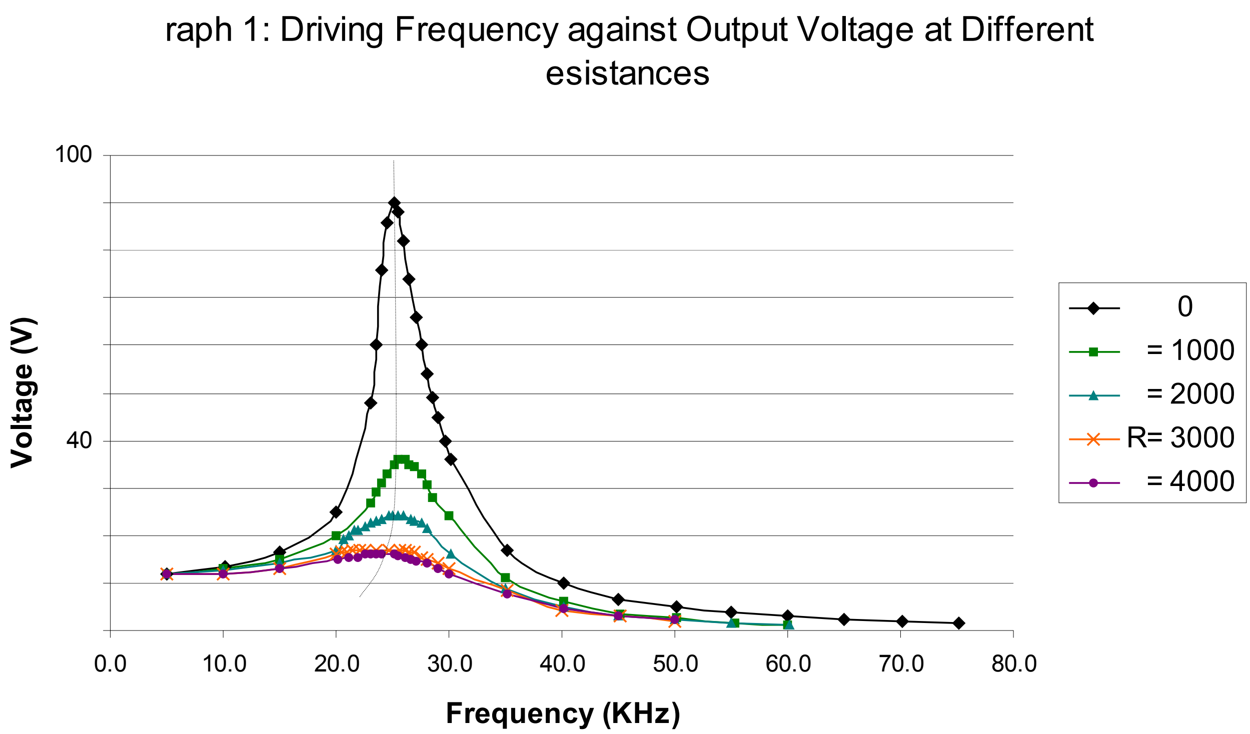
$$f_o = (1/2\pi)\sqrt{(1/LC)} = 29.06 \pm 0.41\ KHz \qquad (11)$$

(Note that formulae for errors dealt with in this investigation are listed in the appendix)

3.2 Experimental Method

The peak to peak amplitude of the voltage across the capacitor was measured at varying driving frequencies around the estimated natural frequency, i.e. 5 – 75 KHz. The frequency was measured using the frequency meter connected across the oscillator. The voltage is taken from the peak to peak measurement of the oscilloscope connected across the capacitor. As the frequency of the oscillator approached the natural frequency of the circuit, more results were taken to define the natural frequency with higher accuracy. A set of data was taken for values of $R = 0\Omega, 1000\Omega, 2000\Omega, 3000\Omega, 4000\Omega$.
A graph of driving frequency against output voltage was plotted for each value of R (see graph 1).

3.3 Experimental Results



3.4 Analysis of Results – Determination of the Q Factor

From graph 1 it can be seen that the resonance curves get sharper as the resistance decreases. The sharpness of the curve can be quantified by the Q factor as described in section 2.3. Measurements and calculations of the measured Q factor, Q_m, the theoretical value Q_t and all corresponding uncertainties can be found in table 2 of the appendix.

The values of Q_m and Q_t are not seen to be consistent with each other. The discrepancies are noticeably less at higher resistances, this suggests an unmeasured resistance or alternative energy loss in the circuit. In order to correct for this unmeasured resistance, its magnitude was found. This was done by plotting a graph of $1 / V_c^{(res)}$ against R. From equation 7 it can be seen that:

$$\boldsymbol{R}_{total} \propto \boldsymbol{1 / V_c^{(res)}} \qquad (12)$$

Where in practice:

$$\boldsymbol{R}_{total} = \boldsymbol{R} + \boldsymbol{R}_{missing} \qquad (13)$$

Therefore:

$$\boldsymbol{R = k / V_c^{(res)}} - \boldsymbol{R}_{missing} \qquad (14)$$

Where k is the constant of proportionality.

Hence it was expected that the graph of $1 / V_c^{(res)}$ against R would be a straight line with negative x-axis intercept equal to the missing resistance. A computer program was used to calculate a least squares fit of the data, it gave the R-axis intercept or $\boldsymbol{R_{missing}}$ as:

$$\boldsymbol{R_{missing} = 720 \pm 70\ \Omega} \qquad (15)$$

The values of Q_t were recalculated using equation 13 for R_{total} rather than just R. These can be seen in the last column of table 2 in the appendix.

The corrected values of Q_t are very consistent with the measured Q values, all apart from the $R=3000\Omega$ *and* $R=4000\Omega$ results are within accepted error boundaries.

3.5 Uncertainty Analysis

The major source of uncertainty in this experiment is the unmeasured resistance and although its magnitude had been determined, its origin is still unknown. It was first thought that the major contributor to this was the resistance in the oscillator, also the resistance in the wires and the inductor would have added to the unmeasured resistance. These factors were measured. To measure the resistance of the oscillator the amplitude of the voltage across the oscillator was measured on its own. Then a resistance was introduced in parallel with the oscilloscope so that the amplitude of the voltage was decreased to half its original value. This additional resistance is the resistance of the oscillator (R_{osc}). This was measured to be:

$$R_{osc} = 75 \pm 10\ \Omega \qquad (16)$$

The resistances of the wires (R_w) and the inductor (R_l) were measured using a multimeter:

$$R_w = 0.8\ \Omega \qquad (17)$$

$$R_l = 75\ \Omega \qquad (18)$$

It was obvious that these results did not match up with the unmeasured resistance at all, hence there must have be another source of energy loss in the circuit. After further consideration of the circuit, the inductor coil was thought to be the source of this energy loss through the process of *magnetic hysteresis*. As the current coil is forced back and forth by the AC supply, a small amount of energy is lost each cycle. This is due to the induced magnetic field, being forced to change direction with current. This causes eddy currents in the core which then causes heating. Since the frequencies used are so high, the effect of this is that a lot of energy is being lost per second.

4 Determination of Phase Angle

4.1 Apparatus Set-up

The relative phase of the voltage across the capacitor with respect to the output voltage of the oscillator can be determined by displaying both signals on the oscilloscope simultaneously. It can be set up so that the oscillator voltage (E_o) determines the deflection in the x direction and the voltage across the capacitor (V_c) determines the y deflection. The result of this is a Lissajous ellipse. The ellipse is formed by the oscilloscope spot being driven in simple harmonic motion in both directions at the same frequency.

4.2 Experimental Method

The phase difference between the two sources can be determined from the measurements of the ellipse (see diagram 4).

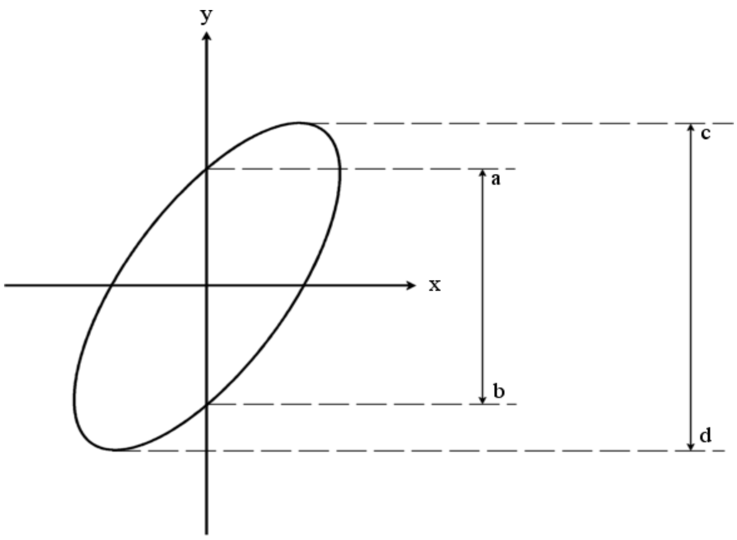


Diagram 4: The Lissajous

In diagram 4, ab is the distance between the y intercepts and cd is the distance between the max and min points. The phase angle ϕ is related to ab and cd by:

$$\sin \phi = ab/cd \tag{19}$$

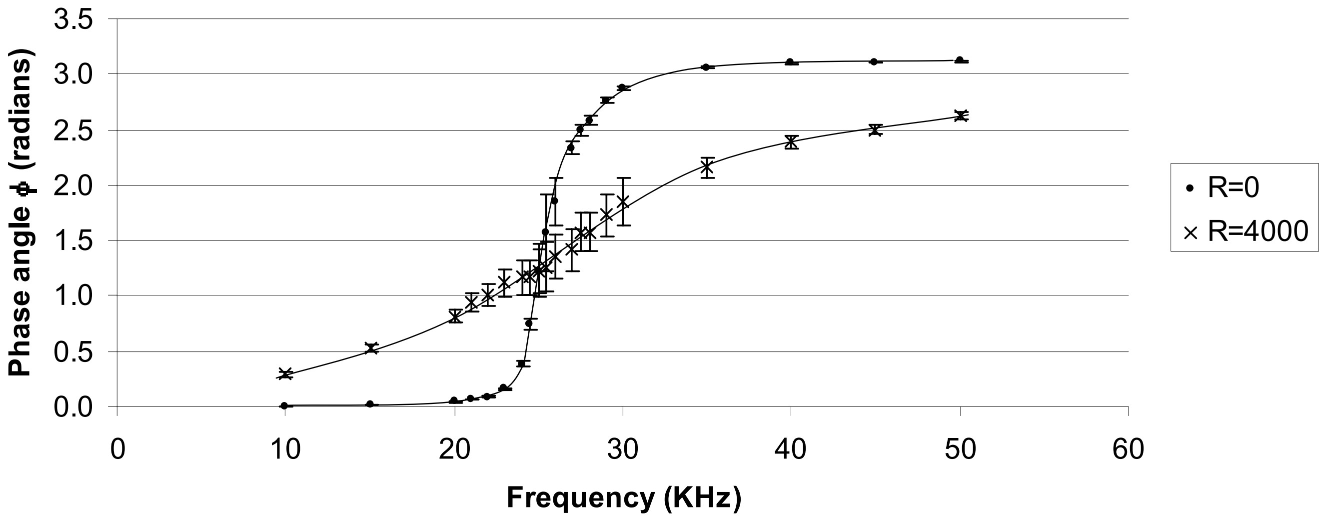
If the ellipse was inclined to the right then $0 < |\phi| < \pi/2$. If it was inclined to the left then $\pi/2 < |\phi| < \pi$. A vertical or horizontal ellipse or a circle indicated that $|\phi| = \pi/2$. A straight line inclined to the right indicated $\phi = 0$. A straight line inclined to the left indicated $\phi = \pm\pi$.

Measurements of ab and cd were taken while the frequency of the oscillator was varied from $10KHz$ to $50KHz$. This was carried out for both low damping

($R=0\Omega$) and high damping ($R=4000\Omega$). Graphs of frequency and phase angle were plotted for low and high damping, along with the uncertainties associated with the measurements (see graph 2).

4.3 Experimental Results

Graph 2: Phase angle against driving frequency for low and high damping



The errors seen in this graph arise from the errors in reading the oscilloscope. See section 6.22 for the formulae used.

4.4 Analysis of Results

The shape of graph 2 is consistent with the theory introduced in section 2.2. It can be seen that for both low and high damping, the phase difference tends to zero when $f << f_o$; equals $\pi/2$ when $f \approx f_o$ and tends to π when $f >> f_o$.

When analysed more quantitatively $\phi = \pi/2$ at $f = 25.5$ KHz for the $R=0\Omega$ case, which is very consistent with the previous result of $f_o = 25$ KHz. For the $R=4000\Omega$ case, $\phi = \pi/2$ at $f = 27.5$ KHz which is quite a bit larger than the previous result. But since the errors in this region are so high, the discrepancy in this value of f_o is accounted for.

The general shape of the curves is consistent with theory because a larger damping effect will tend to slow the response of the capacitor, therefore the signal across the capacitor will lag behind the output signal more, hence the phase graph will be more stretched out, as seen in graph 2.

5 Conclusion

Looking at the final results of the Q factor (table 2 in appendix) it is clear that the measured values are fairly precise. The errors in the measured value mostly due to the uncertainty in reading the graphs. More points plotted around the $V_c^{(res)}/\sqrt{2}$ area in graph 1 would have led to a more accurate measurement of the half power width Δf. This therefore would have affected the value of Q_m. Also the half power height of the $R=4000\Omega$ curve was calculated at less than the starting height of the curve and therefore had to be taken as 12V. This led to a lower value of Δf, which in turn led to a greater value of Q_m than expected. But with these errors aside, the general trend that the Q factor decreases as damping increases can easily be seen here. Also this experiment has shown the relationships of amplitude resonance and phase difference as functions of damping for LCR circutis.

Further refinements to this experiment would be to confirm that the energy loss in the inductor is indeed the major contributor to the unmeasured resistance mentioned in sections 3.4 and 3.5. This could be done by plotting a hysteresis curve of magnetic field strength over a cycle. The energy loss can be estimated from the area under the curve. This is very tricky due to two factors: (i) The period of oscillation in this case is very small so a computer would be needed to take measurements of the magnetic field strength. (ii) The magnitude of the induced magnetic field is very small hence very sensitive equipment would be needed. One way round this would be to place the inductor in a heat bath and measure the change in temperature of the water over a certain time, the corresponding energy change can then be calculated. This then could be compared to the energy loss predicted earlier.

6 Appendix

6.1 Tables of data

Table 1: Comparison of calculated and experimental maximum frequency for different damping resistances.

Resistance (Ω)	Calculated max freq (KHz) (f_{max})	Experimental max freq (KHz)
0	29.1 ± 0.4	25.0 ± 0.5
1000	28.8 ± 0.4	25.5 ± 1
2000	28.1 ± 0.4	25.0 ± 1
3000	26.8 ± 0.4	24.5 ± 2
4000	24.9 ± 0.4	24.0 ± 2

Table 2: Determination of Q factor from graph 1 (Q_m) and calculated theoretical values from equation 10 (Q_t).

R (Ω)	$V_c^{(res)}/\sqrt{2}$ (V)	Δf (KHz)	$Q_m \pm \Delta Q_m$	$Q_t \pm \Delta Q_t$	Q_t corrected
0	63.6	3 ± 1	8.33 ± 2.78	∞	7.61 ± 0.75
1000	25.5	7.5 ± 1	3.40 ± 0.47	5.48 ± 0.12	3.18 ± 0.14
2000	17.0	11 ± 1	2.27 ± 0.23	2.74 ± 0.06	2.01 ± 0.06
3000	12.6	26 ± 1	0.94 ± 0.08	1.83 ± 0.04	1.47 ± 0.03
4000	11.3	25.5 ± 2	0.94 ± 0.11	1.37 ± 0.03	1.16 ± 0.02

6.2 Error Formulae

(note that δ denotes the error on a value)

6.2.1 Formulae for section 3.1

From equation 5:

$$\omega_o = \sqrt{1/LC}$$

Hence:

$$\delta\,\omega_o = \omega_o \sqrt{\tfrac{1}{4}(\delta L/L)^2 + \tfrac{1}{4}(\delta C/C)^2}$$

6.2.2 Formulae for section 4.3

From equation 19:

$$\phi = \sin^{-1}(ab/cd)$$

If we let $ab/cd = x$ then:

$$\delta\phi/\delta x = 1/\sqrt{(1- x)^2}$$

Where:

$$\delta x = (ab/cd)\sqrt{((\delta\, ab / ab)^2 + (\delta cd / cd)^2)}$$

6.2.3 Formulae for table 1 in section 6.1

From equation 6:

$$\omega_{max}^2 = \omega_o^2 - R^2/2L^2$$

Hence:

$$\frac{\delta\,\omega_{max}}{\omega_{max}} = \frac{\sqrt{(4\omega_o^2\,\delta\omega_o^2 + ((R^2\,\delta R^2) / L^4) + ((R^4\,\delta L^2) / L^6))}}{2\omega_o^2 - (R^2 / L^2)}$$

And from equation 2:

$$\delta f_{max} = \delta\omega_{max} / 2\pi$$

6.2.4 Formulae for table 2 in section 6.1

From equation 10:

$$Q_t = (1/R)\sqrt{(L/C)}$$

Hence:

$$\delta Q_t = Q_t\sqrt{((\delta R/R)^2 + \tfrac{1}{4}(\delta L/L)^2 + \tfrac{1}{4}(\delta C/C)^2)}$$

From equation 8:

$$Q_m = f/\Delta f$$

Hence:

$$\delta Q_m = Q_m\ \sqrt{((\delta f / f)^2 + (\delta\Delta f / \Delta f)^2)}$$